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THE PROBLEM OF A GROWING ICICLE*

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One of the applications of the mechanics of growing viscoelastic bodies, associated with the determination of the stressed-deformed state and regularities in the shape formation of a growing or thawing icicle, is considered.

The growth of icicles is an exceedingly complex physico-mechanical process, the detailed investigation of which runs into very considerable difficulties. A mechanical model of an icicle is proposed in this paper which enables one, in our opinion, to work out a technqiue for estimating its stressed-deformed state and to predict the time of its destruction, that is, its detachment. This model makes no pretence whatsoever to being complete and must be considered as a first approximation when solving this problem. It is quite possible that this problem will enable one to develop an approach to the solution of appropriate technological and glaciological problems.

1. Formulation of the problem of the growth of an icicle. Basic relationships.

Let us confine ourselves to the assumption that an icicle is a drawn out figure of rotation, the length of which is significantly greater than the maximum diameter of the cross-sectional area. Let us suppose that the icicle is fixed to a planar horizontal surface which is perpendicular to its longitudinal axis, and let us place the origin of the cylindrical coordinate system (r, φ, z) in the cross-section clamped to the surface with the z-axis directed downwards along the axis of the icicle (Fig.1). It is further assumed that the growth of the icicle starts off from a certain initial configuration which, in principle, may be taken as the shape of a frozen suspended drop. Let l_0 and $R_0(z)$ be the length and radius of the transverse cross-section of the icicle at the initial instant of time $t = 0$; $l(t)$ and $R(t, z)$ the overall length of the icicle and the current radius of its transverse cross-section, and $R(t, l(t)) \equiv 0$.

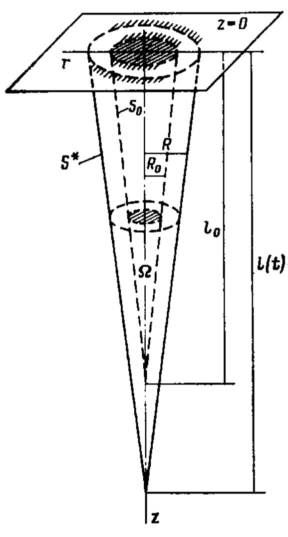

Fig.1

Let us surface S_0 of the initial icicle in the coordinate system $(r, z) = \mathbf{x}$ be specified in a parametric or doordinate form

$$S_0 = \{\mathbf{x}: \mathbf{x} = \mathbf{x}_0(\tau), \ \tau \in [\tau_0, \tau_1]\} \tag{1.1}$$
$$S_0 = \{\mathbf{x}: F_0(r, z) \equiv r - R_0(z) = 0, \ z \in [0, l_0]\}$$

where τ_0, τ_1 are the value of a certain parameter τ (which may, for example, be the length of an arc) which correspond to points on the contour of the axial cross-section of the icicle when $z = 0$ and $z = l_0$.

The boundary of the ice $S^* = S^*(t)$, which is changing as the icicle grows, is similarly specified using one of the following methods:

$$S^*(t) = \{\mathbf{x}: \mathbf{x} = \mathbf{x}^*(t, \tau), \ \tau \in [\tau_0, \tau_1(t)], \ t \geqslant 0\} \tag{1.2}$$
$$S^*(t) = \{\mathbf{x}: F(t, r, z) \equiv r - R(t, z) = 0, \ z \in [0, l(t)], \ t \geqslant 0\}$$

Let us assume that the surface $S^*(t)$ remains quite tapered during any stage in the growth of the icicle, that is,

$$|\partial F/\partial z| \ll |\partial F/\partial r|, \quad F = 0, \quad t \geqslant 0 \tag{1.3}$$

This means that $|\partial R/\partial z| \ll 1$, that is, the angle between the tangent to the contour of the axial cross-section of the icicle and the z-axis is extremely small. The tapering condition (1.3) enables us to assume that the stressed state of the elongated body under

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consideration is close to uniaxial and all the components of the stress are negligibly small compared with the normal **axial stress** $\sigma_z \equiv \sigma$.

The quasistatic equilibrium of the icicle is ensured by the balance of the normal stresses which are distributed in an arbitrary transverse cross-section and the weight of that part of the icicle located below this cross-section. The equilibrium equation has the form

$$2\pi \int_0^{R(t,z)} \sigma(t,z,r)\, r\, dr = \pi g \rho_s \int_z^{l(t)} R^2(t,\zeta)\, d\zeta \tag{1.4}$$

where ρ_s is a constant (the density of ice). In this equation no account is taken of the thin layer of water which flows over the surface of the icicle. We shall neglect the effect of the weight of the aqueous film which is flowing off and of other hydromechanical effects on the stressed state of the icicle.

Let us introuduce the function $\varepsilon_0(t,z)$ into the treatment. This function is equal to the axial deformation of the material at points on the z-axis. We shall denote the axial deformation at an arbitrary point bulk by $\varepsilon(t,z,r)$. In accordance with the general formulation of an initial-boundary value problem for a growing body /1/, the compatibility conditions are, in general, not satisfied but the rates of the deformations do satisfy the compatibility condition (a time derivative is denoted by a dot):

$$\varepsilon^{\cdot}(t,z,r) = \varepsilon_0^{\cdot}(t,z) \tag{1.5}$$

We shall assume that an arbitrary infinitely small volume of the icicle, located in the neighbourhood of a point with coordinates (r,z) begins its existence from the moment of its genesis $\tau^*(r,z)$ (the moment when the water freezes at this point). In the case of points within the initial icicle, this moment of formation is assumed to be equal to zero whereas, in the case of points lying in a domain where there is growth, it is obvious that $\tau^*(r,z) > 0$.

By integrating Eq.(1.5) with respect to time from the moment of the addition of a volume element $\tau^*(r,z)$ up to a certain current value of t, we shall have

$$\varepsilon(t,z,r) - \varepsilon^*(r,z) = \varepsilon_0(t,z) - \varepsilon_0(\tau^*(r,z),z) \tag{1.6}$$

where $\varepsilon^*(r,z)$ is the initial axial deformation which arises at the moment of the addition of the element to the icicle and is associated, for example, with the "volume defect" of the water on freezing. Relationship (1.6) reflects the fact that the elements which have solidified are in a deformed state, which is incompatible with the state of the main body. Apart from depending on the initial deformation ε^*, the stresses in these solidified elements will only be dependent on the growth of the axial deformation of the icicle after the moment of attachment. The axial deformation on the end of the icicle is obviously equal to zero:

$$\varepsilon_0(t, l(t)) = 0 \tag{1.7}$$

In order to describe the mechanical behaviour of the ice under uniaxial tension we shall make use of the characteristic relationship in the non-linear theory of creep

$$\frac{d\varepsilon}{dt} = \frac{1}{E}\frac{d\sigma}{dt} + K\sigma^n; \quad E = \text{const}, \quad K > 0, \quad n > 1 \tag{1.8}$$

where E is the instantaneous modulus of elasticity and K and n are constants which characterize the creep of the ice.

We shall subsequently analyse two limiting cases when the first or second term on the right-hand side of Eq.(1.8) is the predominant term.

2. Rapid growth of an icicle. Let us assume that the growth of the icicle is quite rapid so that the effects due to the creep of the ice do not have time to manifest themselves in any noticeable way. In this case the defining Eq.(1.8) takes the form of Hooke's law

$$\varepsilon = \sigma/E \tag{2.1}$$

Taking into account relationships (1.6) and (2.1), we shall rewrite the equation for the equilibrium of the icicle (1.4) in the form

$$\begin{gathered}\frac{1}{2}R^2(t,z)\varepsilon_0(t,z) - \int_{R_0(z)}^{R(t,z)} \varepsilon_0(\tau^*(r,z),z)\, r\, dr = \\ \frac{\beta}{2}\int_z^{l(t)} R^2(t,\zeta)\, d\zeta - \int_{R_0(z)}^{R(t,z)} \varepsilon^*(r)\, r\, dr, \quad \beta = \frac{g\rho_s}{E}\end{gathered} \tag{2.2}$$

Let us change the variable of integration $\tau^*(r,z) = \xi$ in the second term of the left-hand side of (2.2), allowing for the fact that the z-coordinate is fixed, and differentiate both sides of inequality (2.2) with respect to t

$$\frac{1}{2} R^2(t, z)\,\varepsilon_0^{\cdot}(t, z) = \beta \int_z^{l(t)} R(t, \zeta) R^{\cdot}(t, \zeta)\, d\zeta - \varepsilon^*(R(t, z))\, R(t, z)\, R^{\cdot}(t, z) \tag{2.3}$$

The instant of time $\tau_1^*(z)$ at which the overall length of the icicle is equal to z is associated with the function $\tau^*(r, z)$ introduced earlier by the relationship $\tau_1^*(z) = \tau^*(0, z)$. It is easy to see that the functions $l(t)$ and $\tau_1^*(z)$ are mutually inverse:

$$\tau_1^*(l(t)) \equiv t, \quad l(\tau_1^*(z)) \equiv z \tag{2.4}$$

Let us divide both sides of Eq.(2.3) by $^1/_2 R^2(t, z)$, integrate the resulting equation from the instant of formation of the cross-section $z = \text{const}$ up to a certain current instant in time and change the order of integration with respect to ζ and τ. Finally, we shall have

$$\varepsilon_0(t, z) = \varepsilon^{\circ}(z) + 2\beta \int_z^{l(t)} \Big[\int_{\tau_1^*(\zeta)}^{t} \frac{R(\tau, \zeta)}{R^2(\tau, z)} \frac{\partial R(\tau, \zeta)}{\partial \tau}\, d\tau \Big] d\zeta - 2 \int_{\tau_1^*(z)}^{t} \frac{\varepsilon^*(R(\tau, z))}{R(\tau, z)} \frac{\partial R(\tau, z)}{\partial \tau}\, d\tau \tag{2.5}$$

The quantity $\varepsilon^{\circ}(z) = \varepsilon_0(\tau_1^*(z), z)$ in the resulting expression is equal to the deformation of the axis of the icicle at a point $z = \text{const}$ at the instant of the formation of the corresponding cross-section. It is obvious that the magnitude of $\varepsilon^{\circ}(z)$ may be put equal to zero in the case of the region of growth $z \geqslant l_0$ (see Eq.(1.7)) taking relationships (2.4) into account) while, in the case of the domain $0 \leqslant z \leqslant l_0$, it is found from the equilibrium equation at the instant of the start of growth

$$\varepsilon^{\circ}(z) = \frac{\beta}{R_0^2(z)} \int_z^{l_0} R_0^2(\zeta)\, d\zeta \tag{2.6}$$

and is completely determined by the configuration of the initial icicle.

Hence, expression (2.5) provides a solution to the problem of the formation of the stressed-deformed state of an elastic axially symmetric icicle. The functions $R(t, z)$ and $l(t)$ which define the law governing the formation of the shape of the icicle have been assumed to be specified up to now. They will be defined below in Sect.5. The function $\varepsilon^*(z)$ in (2.5) which determines the initial deformation of the solidifying film of liquid is also assumed to be specified. It may be put equal to zero in a number of cases.

3. The slow growth of an icicle. Let us now consider another limiting case when the growth of the icicle is so slow that the rate of change of the stresses can be neglected. In this case the decisive equation has the form of Glen's law.

$$\varepsilon^{\cdot} = K\sigma^n \tag{3.1}$$

where K and n are constants which characterize the creep of the ice over a specified range of changes in the temperature and other parameters of the surrounding medium. The inverse of this equation has the form

$$\sigma = (K^{-1}\varepsilon^{\cdot})^{1/n} \tag{3.2}$$

Carrying out the transformations of the equilibrium Eq.(1.4) for the given case, we obtain

$$\int_0^{R(t,z)} \sigma(t, z)\, r\, dr = \frac{g\rho_s}{2} \int_z^{l(t)} R^2(t, \zeta)\, d\zeta \tag{3.3}$$

Here, it has been taken into account that the rate of axial deformation (and, consequently, also the values of the stresses) of the initial core of the icicle and of the elements of the growth zone in the transverse cross-section being considered are equal in accordance with relationship (1.5). By substituting expression (3.2) into (3.3), we shall have

$$\frac{d\varepsilon(t, z)}{dt} = \gamma \Big[\frac{1}{R^2(t, z)} \int_z^{l(t)} R^2(t, \zeta)\, d\zeta \Big]^n, \quad \gamma = K(g\rho_s)^n = \text{const} \tag{3.4}$$

Expression (3.4) determines the behaviour of the growing icicle under conditions of non-linear creep which is described by the equation of state (3.1). At the same time, as in the case of the rapid growth of the icicle, the function $R(t, z)$ and $l(t)$ are to be determined.

4. Criteria for the breakdown of the icicle. One of the basic questions which arise in connection with the treatment of the problem of the growth of an icicle concerns the prediction of the moment when it will break and become detached under the action of its own weight. Let us first consider the case of an elastic icicle, the stressed-deformed state of which is determined by relationships (2.1), (2.5) and (2.6). The criterion of maximum permissible stresses /3, 4/ is conventionally employed as the condition for the strength and the brittle breakdown of ice in a state of elastic deformation. We shall denote the magnitude of the permissible stress during the elastic extension of an ice sample by σ_*. The condition for the local brittle fracture of an icicle occupying a domain Ω^* at the instant of breakdown has the form

$$\sigma = \sigma_*, \quad \mathbf{x} \in \Omega^* \tag{4.1}$$

Under the conditions of the problem being considered, it is also necessary to envisage the possibility that the icicle may break away from its support over a region of the area over which it is fixed to the undeformable surface of this support. The strength of the adhesive bond between the ice and a different kind of material, measured in stress units, is usually referred to as the freezing-together force and is substantially dependent on various parameters and, in particular, the physico-mechanical properties of the material supporting the icicle, the irregularities on the surface where the two materials freeze together, the temperature of the surrounding medium, etc. An analysis of the experimental values of the force of the freezing together of ice with certain widely used constructional materials (concrete or timber, for example) at an ambient temperature of 0 to -5°C leads to the conclusion that the force of freezing-together is, as a rule, somewhat less then the tensile strength of ice) at the same temperature and under strictly identical conditions) and that they have approximately the same values, apart from the scatter in the experimental data /3, 4/.

Denoting the force of the freezing-together of the ice and the support surface by σ_{**}, the condition for local breakdown at points in the cross-section where the ice is fixed to the support may be written in the form

$$\sigma = \sigma_{**}, \quad \mathbf{x} \in S^\circ \tag{4.2}$$

where S° is the domain of the clamped cross-section $(z = 0)$ at the instant of breakdown. If the axial stresses in the transverse cross-sectional area of the ice rod and the normal stresses over the area over which it is fixed to the support are uniformly distributed, condition (4.1) or (4.2) will serve as a criterion not only of local breakdown but also as the condition for the partial or complete detachment of the icicle.

However, according to the accepted deformation model, the stresses in the case of a growing icicle are only, generally speaking, uniform within the limits of the initial transverse cross-sectional area of the icicle. On the whole, the distribution of the stresses in a cross-section of a growing icicle is non-uniform since the stresses in the growth zone are formed independently of the state of the main body of the icicle. Conditions (4.1) and (4.2) therefore cannot be used as criteria for the global breakdown or detachemnt of the icicle. Nevertheless, fracture of the initial cross-section results in the state of the icicle becoming dangerous from the point of view of its strength.

Taking into account the fact that, in estimating the strength of an icicle, it is obviously advisable in certain cases to start off from the lower limiting value of the gravitational load, we shall assume that the attainment of the maximum permissible stress level denotes the fracture of the corresponding cross-section and the detachment of that part of the icicle which is located below this cross-section. We shall denote by t_Ω and t_0 the instants of time at which the limiting values of the stresses are attained in accordance with the equations

$$\max_{\mathbf{x} \in \Omega^\circ} \sigma(t_\Omega, \mathbf{x}) = \sigma_*, \quad \max_{\mathbf{x} \in S^\circ} \sigma(t_0, \mathbf{x}) = \sigma_{**} \tag{4.3}$$

The icicle therefore becomes detached at the instant of time $t^* = \min(t_\Omega, t_0)$. If the calculated values of t_0 and t_Ω, defined according to criteria (4.1) and (4.2) on the basis of relationships (2.1), (2.5) and (2.6) are related by the inequality $t_0 < t_\Omega$, the icicle becomes completely detached from the surface to which it was fixed; if $t_\Omega < t_0$ partial detachment occurs at the fractured section. As regards the case of the purely viscous state of an icicle, which is defined by relationships (3.1) and (3.4), in view of the wide variety of strength and fracture criteria for materials undergoing creep, there are certain difficulties associated with the choice of an actual criterion which is applicable to the conditions of the problem under consideration. Obviously, the criterion of a limiting deformation rate (see /3/): $d\varepsilon/dt = e_*$, where e_* is the critical deformation rate at which the sample fractures, is the simplest criterion for the fracture of ice which reflects the special features of its deformation under conditions of creep. Since, by virtue of (1.5), the rates of deformation of the core of the icicle and of the cross-section of elements which are added to it are equal, the criteria for local and global fracture are identical and are expressed by the equation $\varepsilon_0^{\cdot} = e_*$, where $\varepsilon_0^{\cdot}$ is the rate of deformation of the axis of the icicle.

5. The shaping of a growing icicle. It has already been mentioned that the formulae obtained above, (2.5) and (3.4), are a solution of the problem on the build up of the stressed state of an icicle with the reservation that the mechanism of the change in its geometrical shape, which is determined by the conditions for heat and mass transfer with the surrounding medium, are known. The description of the heat and mass exchange between the body of the icicle and the water flowing off over it, as a result of the partial or complete freezing of which the icicle grows, is a separate and, most likely, a no less complex problem than the analysis of the stressed-deformed state of the icicle. One of the possible approaches to the solution of this problem is proposed below.

5.1. The phase transition problem for a growing icicle. Strictly speaking, it is necessary to consider the overall mathematical formulation of the problem of a phase transition in order to analyse the motion of the interphase boundary as the water freezes during the growth of the icicle. A two-dimensional two-phase Stefan problem with a free boundary is one of the possible formulations. However, there are quite considerable difficulties in obtaining a solution of this problem in a clear form. We shall therefore simplify the formulation of the problem by confining ourselves to a treatment of the one-dimensional Stefan problem in a local coordinate system (ν, s) when the "water-ice" phase transition front moves along the ν coordinate, which is measured off along the external normal to the surface of the initial icicle (by virtue of condition (1.3), the normal is, in fact, directed along a radius of the transverse cross-section).

Taking into account the fact that, by virtue of condition (1.3) which has been adopted, the surface of the icicle is gently tapered, we shall confine ourselves to an analysis of the heat exchange between the ice and the water flowing off over it within the framework of the one-dimensional problem for a transverse cross-section of the icicle at an arbitrary fixed coordinate z.

The heat-conduction equation for ice in the cross-section of an icicle has the form

$$\frac{\partial T_s}{\partial t} = c^2\left(\frac{\partial^2 T_s}{\partial r^2} + \frac{1}{r}\frac{\partial T_s}{\partial r}\right), \quad 0 \leqslant r \leqslant R(t, z) \tag{5.1}$$

where T_s is the temperature of the ice and c^2 is the thermal diffusivity of ice. Under the conditions of the problem being considered, the most intense heat exchange occurs in the comparatively thin layer of the ice which is adjacent to the moving boundary of separation of the phases. It can be shown that the first term on the right-hand side of Eq. (5.1) is decisive in the case of this layer. At the same time the front of the phase transition is by now considered to be planar.

The heat-conduction equation for ice and the Stefan condition on the interphase boundary therefore have the form /5/

$$\frac{\partial T_s}{\partial t} = c^2\frac{\partial^2 T_s}{\partial \nu^2}, \quad -\infty < \nu < \xi(t) \tag{5.2}$$

$$k_s\frac{\partial T_s}{\partial \nu}\bigg|_{\nu=\xi(t)-0} - k_l\frac{\partial T_l}{\partial \nu}\bigg|_{\nu=\xi(t)+0} = \lambda\rho_l\frac{d\xi}{dt} \tag{5.3}$$

Here T_l is the temperature of the water, k_s and k_l are the thermal conductivities of the solid and liquid phases respectively, λ is the latent heat of phase transition per unit mass and ρ_l is the density of water. The points $\nu = \xi - 0$ and $\nu = \xi + 0$ are limiting points on the boundary of separation of the phases as this boundary is approached from the side of the ice and the water.

We shall assume that the temperature of the ice when $\nu = -\infty$ (i.e. sufficiently far from the phase transition boundary) is constant and below the phase transition temperature. We shall also assume that the temperature of the water is constant and equal to the freezing point of water. Hence, in addition to relationships (5.2) and (5.3), we have the conditions

$$T_l \equiv 0, \ \ \nu \geqslant \xi(t); \quad T_s = T_l = 0, \ \ \nu = \xi(t); \quad T_s = T_0 < 0, \ \ \nu = -\infty \tag{5.4}$$

Similarly, /5/, it may be shown that the temperature distribution in the ice and the position of the interphase boundary are determined by the formulae

$$T_s(t, \nu) = T_0\left[1 + \operatorname{erf}\left(\frac{\alpha}{2c}\right)\right]^{-1}\left[\operatorname{erf}\left(\frac{\alpha}{2c}\right) - \operatorname{erf}\left(\frac{\nu}{2c\sqrt{t}}\right)\right] \tag{5.5}$$

$$\xi(t) = \alpha\sqrt{t}$$

The constant α is a solution of the following transcendental equation:

$$\left[1+\operatorname{erf}\left(\frac{\alpha}{2c}\right)\right]^{-1}\exp\left(-\frac{\alpha^2}{4c^2}\right)=D\alpha \tag{5.6}$$
$$D=-\sqrt{\pi}\lambda\rho_l c/(2k_s T_0)>0$$

It is obvious that Eq.(5.6) always has a root $\alpha > 0$.

5.2. Kinematics of the shaping of a growing icicle. The second expression in (5.5) determines the thickness of the layer of ice which is joined to the initial surface of the icicle, measured along the external normal to it. If the surface of the initial icicle is specified by relationships (1.1), the surface of the ice in the new position is defined by the equation

$$\mathbf{x}=\mathbf{x}_0+\xi\nabla F_0/|\nabla F_0| \tag{5.7}$$

By virtue of the gentle tapering condition (1.3), the vector Eq.(5.7) can be reduced to the approximate scalar relationship

$$R(t,z)=R_0(z)+\xi(t) \tag{5.8}$$

where the function $\xi(t)$ is defined by relationships (5.5) and (5.6). The resulting approximate expression for the function $R(t,z)$ can be used directly to determine the stressed-deformed state of the icicle using formulae (2.5) of (3.4). We shall define the function $l(t)$ in these formulae, which is equal to the length of the growing icicle, on the basis of the following relationships:

$$l(t)=l_0+\int_0^t\frac{dl(\tau)}{d\tau}d\tau,\quad \frac{dl}{dt}=\frac{l_0}{\max_z R_0(z)}\frac{d\xi}{dt} \tag{5.9}$$

The growth of the icicle in the transverse direction in the interval $l_0 \leqslant z \leqslant l(t)$ is then defined using the formula

$$R(t,z)=\xi(t)-\xi(\tau_1^*(z)) \tag{5.10}$$

where $\tau_1^*(z)$ is the function specified by relationships (2.4).

5.3. Analysis of the results. To be specific, let us assume that the surface of the initial icicle has the shape of a narrow cone

$$r=\mp Az\pm B,\quad A>0,\quad B>0,\quad 0\leqslant z\leqslant B/A\equiv l_0 \tag{5.11}$$

where the coefficient A is quite small.

We shall confine ourselves to the treatment of an elastic icicle, the stressed-deformed state of which is defined by relationships (2.5) and (2.6).

Let us put $\varepsilon^*\equiv 0$ in these relationships. Calculations using formulae (2.5), (2.6), (5.5), (5.8)-(5.11) give the following results:

$$\varepsilon_0(t,z)=\alpha\beta\sqrt{t}/A+{}^1/_3(l_0-z)\beta,\quad 0\leqslant z\leqslant l_0 \tag{5.12}$$
$$\varepsilon_0(t,z)=\alpha\beta\sqrt{t}/A-(z-l_0)\beta,\quad l_0\leqslant z\leqslant l(t)$$

It can be seen from expressions (5.12) that the maximum value of the deformation of the ice is reached in the cross-section $z=0$, that is, on the surface where the ice is frozen to its support. At the same time, the deformation $\varepsilon_0(t,0)$ increases monotonically with time as the icicle grows. Let us assume that the condition for the strength of the icicle is satisfied at the instant when growth starts, $t=0$. Then, from relationships (2.1) and (4.2) and the first expression of (5.12), we obtain the formula for the instant when the icicle will fracture

$$t_0=\left[\frac{A}{\alpha\beta}\left(\frac{\sigma_{**}}{E}-\frac{\beta l_0}{3}\right)\right]^2$$

One important, but not completely obvious, fact should be emphasized: the stressed-deformed state of a growing elastic icicle is quite different from the state of the same icicle with identical geometrical parameters but which has been deformed instantaneously or under such conditions when gravitational forces begin to act upon completion of the build up.

Actually, we shall calculate the elastic axial deformation $\varepsilon_0^*(t,z^*)$ of an icicle which has been instantaneously formed at an instant of time t for a certain fixed cross-section $z^*=\text{const}<l_0$. When this is done, the radius of the transverse cross-section, in accordance with formula (5.8), is equal to $R(t,z^*)=\alpha\sqrt{t}+A(l_0-z)$ while the overall length of the icicle, according to relationships (5.9), is equal to $l(t)=l_0+\alpha\sqrt{t}/A$. We shall have $\varepsilon_0^*(t,z^*)=V_{z*}\rho_s g/(S_{z*}E)$, where $V_{z*}={}^1/_3 S_{z*}(l-z^*)$ is the volume of the ice included in the cone, lying below the cross-section z^* and S_{z*} is the area of this cross-section. Hence,

$$\varepsilon_0^*(t,z^*)={}^1/_3\beta(l_0+\alpha\sqrt{t}/A-z^*)$$

By comparing the resulting value with the corresponding deformation of an icicle $\varepsilon_0(t, z^*)$ which has been deformed over a finite interval of time $[0, t]$ in the gravitational force field (see the first expression of (5.12)), we shall write $\varepsilon_0/\varepsilon_0^* = 3 - 2/(1+\gamma)$, where the quantity $\gamma = \alpha\sqrt{t}/[A(l_0 - z^*)]$ determines the ratio of the thickness of the grown layer of ice to the radius of the initial transverse cross-section at a fixed z^* at the instant of time t being considered. It is seen that $\lim_{\gamma\to\infty}(\varepsilon_0/\varepsilon_0^*) = 3$, that is, the deformation of a growing icicle, when the thickness of the layer of ice which has grown onto it is sufficiently large, is practically three times greater than the deformation of an icicle with the same dimensions which has been instantaneously formed. A similar comparison for the growth interval of the axis of the icicle $l_0 \leqslant z^* \leqslant l(t)$, when the deformation $\varepsilon_0(t, z^*)$ is determined by the second expression in (5.12), yields $\varepsilon_0/\varepsilon_0^* = 3$, that is, there is exactly a threefold excess of deformation at any stage of the growth.

This difference in the elastic deformations of growing and "stationary" icicles is due to the fact that, in the case of a growing icicle, its elements perceive the gravitational force little by little as they are formed. In this case the internal elements of a cross-section turn out to be the most loaded and the longitudinal threads adjacent to the axis of the icicle are subjected to the greatest stretching deformations. The deformations are distributed normally over the cross-section of a stationary icicle since all of its elements are simultaneously loaded.

6. The thawing of an icicle. The shift in the thermal balance during the hours of daylight is responsible for the thawing of the ice and a reduction in the overall volume of an icicle. If the thawing out of the ice during the daylight hours is not very intensive, the icicle continues its existence for a further few days. During the morning hours when the air temperature is low it may grow, only to start thawing after the air has warmed up.

We will now proceed to construct a mechanical model of a thawing icicle. In doing this, we shall make use of Glen's law (3.1) to describe the rheological behaviour during thawing. It is obvious that the formation of stresses and the rates of deformation in this case is, as before, determined by relationship (3.4). Unlike the case of a growing icicle, the functions $R(t, z)$ and $l(t)$ in Eq.(3.4) will be monotonically decreasing functions of time when applied to the situation under consideration.

6.1. The dynamics of the dripping down of the water film. The shaping of a thawing icicle is determined by the melting of the surface layer of ice in a heat exchange process with the thin layer (film) of water which is draining down along the external surface of the ice starting from the upper cross-section of the icicle where it is clamped. The amount of water which drains down the icicle is determined by the conditions under which the ice and snow in the feed reservoir are thawing. Let us settle on a certain initial configuration for the icicle and let the shape of the initial icicle at the initial instant of time $t = 0$ be described by relationships of the type (1.1). Thawing of the ice leads to a state of affairs where the solid surface of the icicle becomes the front of an "ice-water" phase transition which is displaced along the internal normal to the initial surface of the ice. As before, we shall denote the changing boundary of the ice at the current instant of time by $S^*(t)$ (see (1.2)) and assume that conditions (1.3) are satisfied.

Everywhere below we shall assume that the thickness of the water film h is significantly less than the characteristic cross-sectional dimensions of the icicle (the hydrodynamic problem of the flow round a solid of rotation subject to this condition may be considered as a planar problem). Moreoyer, we shall confine ourselves to the simplest model of draining off in which the effect of surface tension on the equilibrium liquid film is not taken into account.

Within the framework of condition (1.3) the flow of the film along the surface S^* may be analysed locally as a flow of a thin layer of a viscous incompressible liquid along an inclined plane. Let us introduce the coordinate system (ζ, η), the axis ζ of which is directed downwards along the surface of the icicle while the axis η is directed along the external normal to it. By formulating the equation for the equilibrium of an infinitely thin layer of liquid under the action of tangential stresses and gravitational forces, as was done in /6/, we shall have

$$-\mu du/d\eta = \rho_l g(\eta - h)\cos\theta \tag{6.1}$$

where θ is the angle of inclination of the tangent to the surface of the icicle from the vertical, u is the velocity of the liquid in the direction of the axis ζ, μ is the coefficient of dynamic viscosity. From (6.1) we find the expression for the flow rate

$$u = (\rho_l g/\mu)\cos\theta\,(h\eta - \eta^2/2) \tag{6.2}$$

Let us now write the condition for the conservation of the volume of the liquid during flow along an inclined plane while, for now, still not taking into account the motion of the boundary of separation of the phases (Fig.2). To higher-order small quantities, we obtain

$$V|_{t+dt} = V|_t + (\langle u\rangle h)|_{\zeta+d\zeta}dt - (\langle u\rangle h)|_\zeta dt + \frac{\partial h}{\partial t}d\xi\, dt \tag{6.3}$$

Here V is an elementary volume of the liquid included between the free surface of the flowing film and the inclined plane and also by a pair of infinitely closely arranged cross-sections $\zeta = \text{const}$ and $\zeta + d\zeta = \text{const}$, $\langle u \rangle$ is the rate of flow averaged over the thickness of the layer.

The increase in the volume of the water due to the partial thawing of the ice, that is, the displacement of the bearing surface through the material particles of the bulk of the solid phase, is taken into account by the relationship

$$V|_{t+dt} = V|_t + v\,d\zeta\,dt \tag{6.4}$$

where $v > 0$ is the rate of displacement of the surface of the ice due to thawing directed along the normal to the surface of the icicle into the solid phase.

Relationships (6.3) and (6.4) lead to the equation

$$\partial h/\partial t + \partial(\langle u \rangle h)/\partial \xi = v \tag{6.5}$$

Using expression (6.2) to calculate the magnitude of $\langle u \rangle$, taking account of the fact that $\cos\theta \approx 1$ by virtue of the smallness of the angle θ, and substituting the result into Eq.(6.5), we shall reduce it to the following form:

$$\partial h/\partial t + \rho_l g \mu^{-1} h^2 \partial h/\partial \zeta = v \tag{6.6}$$

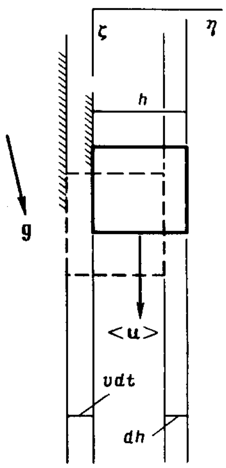

Fig.2

The rate of displacement of the ice surface v must be determined from the additional relationships which describe the heat and mass exchange between the icicle and the aqueous film.

6.2. Heat and mass transfer conditions as a film flows around a thawing icicle. Let us now return to the classical Stefan condition on the boundary of an ice-water phase transition which, when applied to the case under consideration, has the form

$$\lambda\rho_s v = k_l \frac{\partial T_l}{\partial \eta}\Big|_{\eta=+0} - k_s \frac{\partial T_s}{\partial \eta}\Big|_{\eta=-0} \tag{6.7}$$

We shall assume that the temperature of the ice is constant and equal to the melting point of ice $T_s \equiv 0$ and that the temperature of the water within the film is distributed over its thickness in accordance with the linear law

$$T_s = 0, \quad \eta \leqslant 0; \quad T_l = T_a\eta/h, \quad 0 \leqslant \eta \leqslant h \tag{6.8}$$

where T_a is the temperature of the surrounding air $(T_a > 0)$. These assumptions are quite well founded in the case of comparatively massive icicles, if the layer of water draining off them is very thin.

Using expressions (6.8), we shall write condition (6.7) in the form

$$\lambda\rho_s v = k_l T_a/h \tag{6.9}$$

Since all the quantities appearing in this equation, apart from v and h, are assumed to be constant, it follows from it that, at the instant of time being considered, the rate of fusion of the ice is inversely proportional to the thickness of the water film which is draining off over the icicle.

6.3. The basic equation of the dynamics of the draining of water over a thawing icicle and its solution. Eqs.(6.6) and (6.9) form a system of equations from which the law governing the motion of the external boundary of the ice S^* must be determined. By substituting (6.9) into (6.6) and introducing the dimensionless variables $h^* = h/d_0$, $t^* = t/t^\circ$, $\zeta^* = \zeta/d_0$, where d_0 and t° are certain characteristic units of length and time, we obtain the basic equation for the flow of the water film (the asterisks on the dimensionless variables have been omitted):

$$h\frac{\partial h}{\partial t} + ah^3\frac{\partial h}{\partial \zeta} = b, \quad a = \frac{\rho_l g t^\circ d_0}{\mu}, \quad b = \frac{2k_l T_a t^\circ}{\lambda\rho_s d_0^2} \tag{6.10}$$

It is natural to take the initial length of the icicle or its characteristic diameter as the scale d_0. As t° one may specify a quantity which either converts the number a or the number b to unity. The solution of the problem of the thawing of the icicle will only be determined by the ratio of the dimensionless parameters a/b. When this ratio is changed from small to exceedingly large values, either thermal or hydrodynamic effects will predominate respectively.

We note that Eq.(6.10) is obtained on the basis of relationships which describe the local flow of the film in the neighbourhood of an arbitrary point of the surface of the icicle. It can be shown that, on passing from the local coordinates (ζ, η) to the orthogonal curvilinear coordinates (ξ, ν), where ξ is the length of an arc measured along the surface S^* and ν is the distance along the orthogonal curve to the coordinate line ξ, Eq.(6.10) retains its form within the framework of condition (1.3) for the gentle tapering of the surface S^*.

By introducing a new unknown function $f = h^2$, let us rewrite Eq.(6.10) in the form

$$\partial f/\partial t + af\partial f/\partial \zeta = b \tag{6.11}$$

In order to solve this equation, we shall make use of the method of characteristics. The characteristic equations have the form

$$df/ds = b, \quad dt/ds = 1, \quad d\zeta/ds = af \tag{6.12}$$

Here, s is a coordinate measured along the characteristics. For now, we shall take initial conditions of the Cauchy type for the system of Eqs.(6.12) in the form of arbitrary functions of a characteristic coordinate τ measured along the line of the initial conditions:

$$f = f_0(\tau), \quad t = t_0(\tau), \quad \zeta = \zeta_0(\tau) \tag{6.13}$$

From the first and last equations of (6.12), we get

$$f(s, \tau) = bs + f_0(\tau), \quad \zeta(s, t) = {}^1/_2 abs^2 + asf_0(\tau) + \zeta_0(\tau) \tag{6.14}$$

The characteristics, corresponding to the solution (6.14) which has been obtained, have the form of branches of parabolas which are defined by the equations

$$\zeta = \left[\sqrt{\frac{ab}{2}}\,t + \sqrt{\frac{a}{2b}} + f_0(\tau)\right]^2 + \zeta_0(\tau) - \frac{a}{2b} f_0^2(\tau)$$

To be specific, let us specify the following initial conditions:

$$t_0(\tau) = 0, \quad \zeta_0(\tau) = \tau, \quad f_0(\tau) = \beta - \alpha\tau, \quad \alpha > 0, \quad \beta > 0 \tag{6.15}$$

In a physical sense, Cauchy-type initial conditions of this form constitute a boundary condition on the function f which is specified at the initial instant of time $t = 0$ on the surface of the initial icicle.

Eliminating the variable τ from Eqs.(6.14), we shall represent the solution of Eqs.(6.11) under the specified conditions (6.15) in the form

$$f(t, \zeta) = bt + \beta + \alpha(a\alpha t - 1)^{-1}(\zeta - {}^1/_2 abt^2 - a\beta t) \tag{6.16}$$

According to the resulting expression, the dimensionless thickness of the liquid layer $h = \sqrt{f}$ increases with time for all points of the surface of the icicle. In accordance with relationship (6.9), the rate of displacement of the phase boundary where thawing occurs through the mass points of the ice will then decrease montonically with time. We note that this conclusion holds within the framework of the model adopted for the thawing of an icicle which is described by Eq.(6.11) for an initial condition of the form (6.15). Expression (6.16), when account is taken of the relation $h = \sqrt{f}$ in conjunction with relationships which are completely analogous to (5.7) and (5.8), can be directly employed to describe the shaping of a thawing icicle.

As $t \to \infty$, it follows from (6.16) that $f \sim {}^1/_2 bt$, that is, the shaping of the icicle asymptotically evolves onto a regime of uniform melting over the whole of its surface with a rate of function $v \sim f^{-1/2}$ which is independent of the longitudinal coordinate. This regime arises immediately if the initial thickness of the film, covering the icicle, is equal to zero.

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